Theory of Computation

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PRIMES Circle

2023

Intro to State Machines

Have you ever been in an elevator before?

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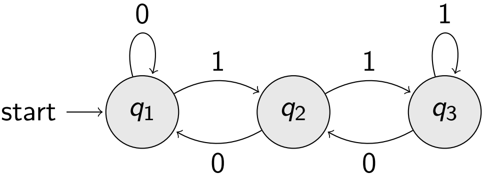

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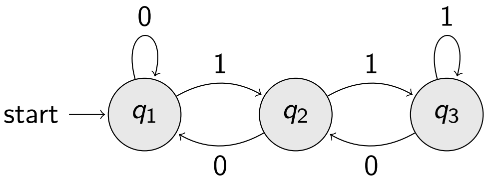

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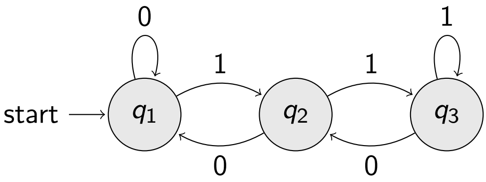
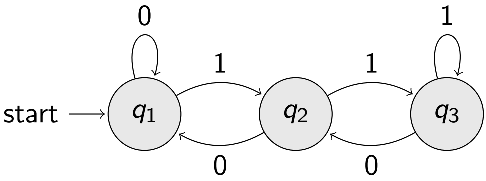


What is that?

This is a state diagram for an elevator in a building with three floors. q_1, q_2, and q_3, which are called *states* represent the first second and third floors of the building, respectively.

State Machine informal definition



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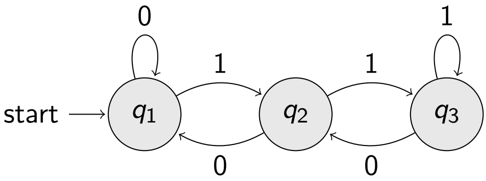

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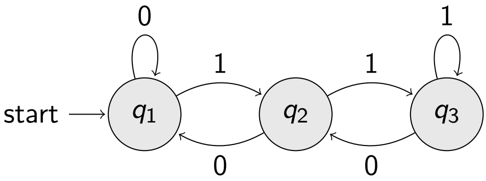

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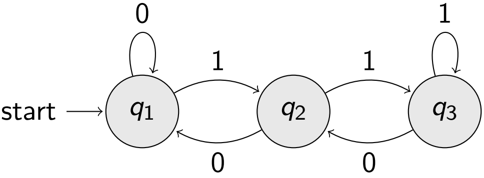

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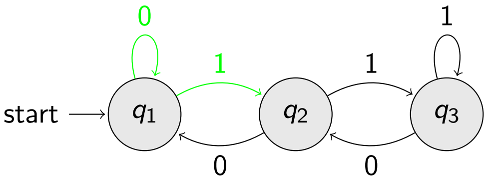

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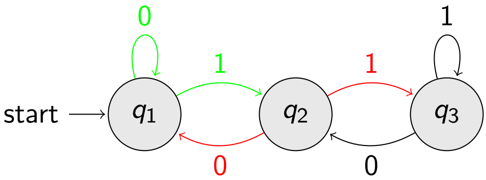

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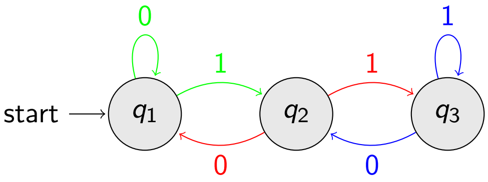

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Input Strings and Accept States

If you were to walk into the elevator and decide to go up, down, up, down, down, and then up, the state diagram would tell you where you would end up after all of that. If you follow the steps in your head, you will end up on the second floor (remember, there is no floor -1).

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By the way, not recommended. If you want exercise, just take the stairs!

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4. In the theory of computation, we want to find if the state machine *rejects* or *accepts* a string. The machine will *accepts* a string if it ends at an accept state, and reject if it does not. Any of the states can be accept states.

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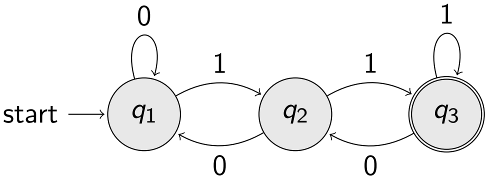

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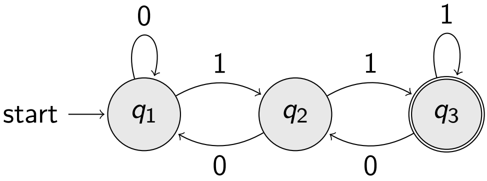
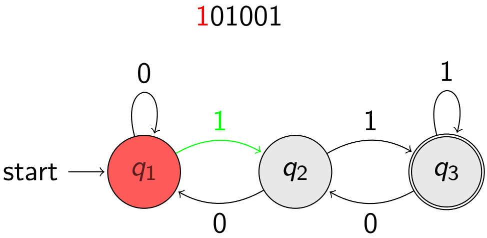


Say we really want to know if we end up on the third floor after all of our movement. This state machine will accept a string if it lands us at state 3.

Input Strings and Accept States

Let's see what this now complete state machine does with our previous input: 101001:



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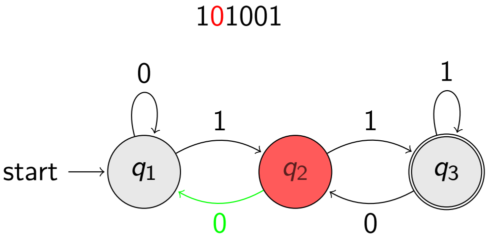

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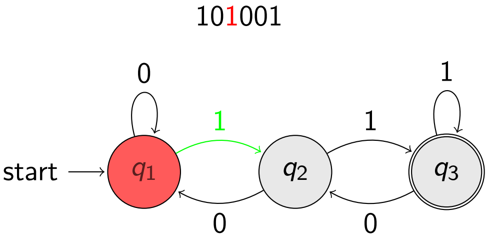

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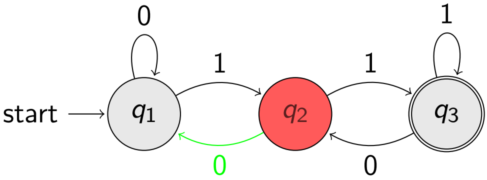

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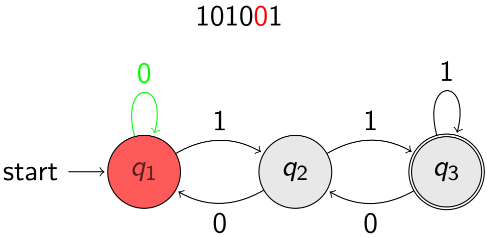

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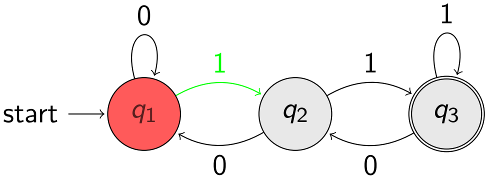

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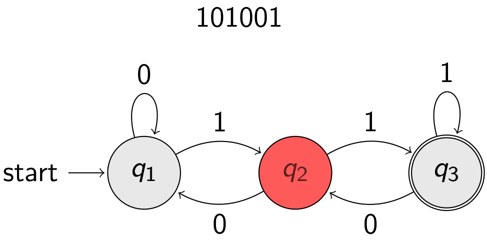

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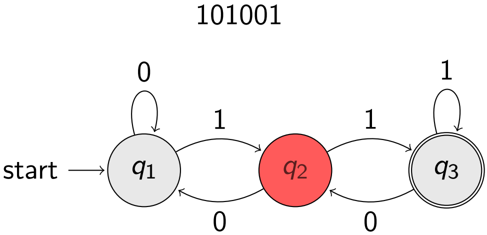


As you can see, the string in our example: 101001, will not be accepted, because it ends at state 2, which is not an accept state.

Languages

Now that we've introduced accept states, let us present the two following definitions.

Definition

The **Language** of a state machine the set of strings that it accepts.

Definition

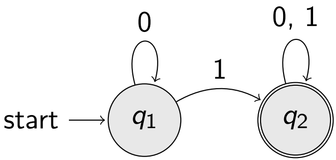
A state machine "recognizes" a language if the set of strings that it accepts is *exactly* equal to the language.

Definition

A set of strings which recognized by *some* DFA is called a **regular language**.

Example

Take this state machine for example:



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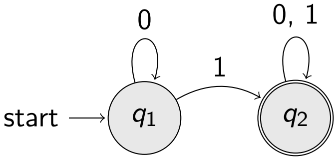

Can you figure out what the language of this state machine is?

It is simply all strings that contain a 1 in them.

1. If a 1 is detected by the machine, it moves to state 2, and stays there, since, at state 2, it loops for both 0 and 1.
2. If there is not a 1 in the string, the machine will loop at state 1, and not accept.

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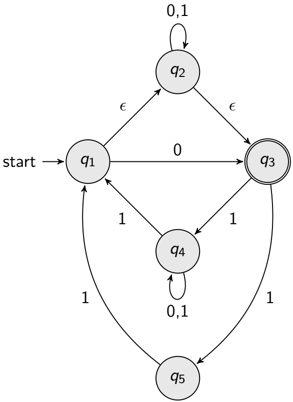
What you have just seen are **Deterministic Finite Automata**. They are called deterministic this third rule makes the state machine take a set path for any input.

NFA Example

The other type of **Finite Automata** is the Nondeterministic finite automata (NFA). Here is an example of the an NFA:



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- We are also allowed a new type of arrow called an ϵ arrow. The way that this arrow works is it transitions without taking in any input symbol, or in other words, taking in the empty symbol.
- Another way to think about it is that for any transition, the NFA goes through all available transition arrows. The ends of these arrows branch out similarly. If any of the final states is an accept state, the NFA accepts.

DFA vs. NFA

NFAs seem much more powerful than their counterpart, because of all of the extra possibilities allowed. Yet, it can be proven that any NFA has an equivalent DFA, or in other words, any language that is recognized by some NFA can also be recognized by some NFA.

DFA/NFA Equivalence

Here is a rough sketch of this proof:
The NFA can be viewed as a *deterministic* machine from sets of states. At any point in computation, the machine is in a set of states, and from each of these states, after undergoing every transition, we end up in another set.

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From an NFA, N, we can create DFA D, where D's states are the sets of the states in N, and D's accept states are any set which contains an accept state of N.

Introducing Turing Machine

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Do you use a computer? Well then you're in luck! I sure do! Turing machines happen to be the first version of modern day computers! Here's what the computers which you now know and love would have looked like all those years ago (all the way back in 1936)

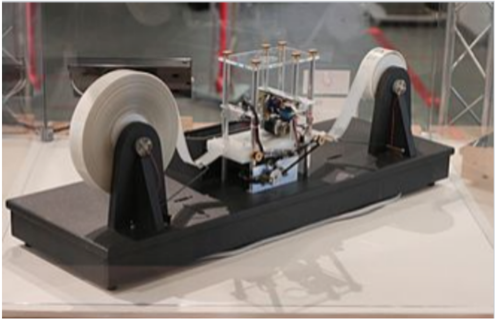

Figure: Turing Machine in 1936

Church-Turing Thesis

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Any real-world computation can be translated into an equivalent problem involving a Turing Machine

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Any computation you could do, a Turing Machine could as well!

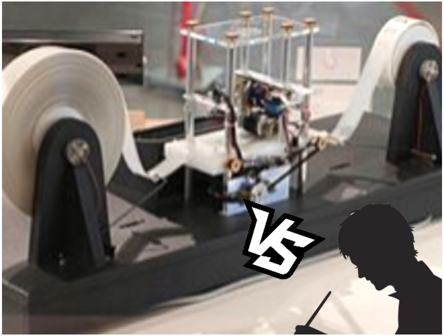

Figure: You vs a Turing Machine

Turing Machine Definition

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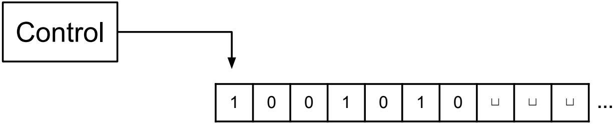
5. During a transition can add or remove a symbol in current location
6. Has only one accept state and reject state

A Closer Look

Let's take a closer look!



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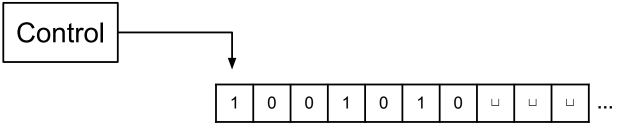


Figure: Example Turing Machine

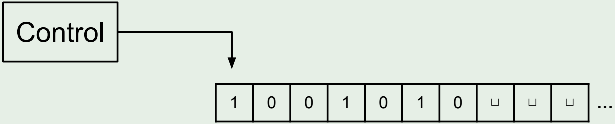
The Control block symbolizes where the transition functions and states are held, while the arrow represents the tape head, pointing to which character of the input string it is at. Finally, we can see the tape holding the string "1001010," the $\sqcup$ symbol telling us that there is an empty space.

Note the "..." is showing us that the tape is infinite

Example

Example

Find if input has equal number of 1's and 0's



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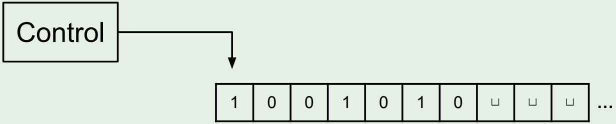

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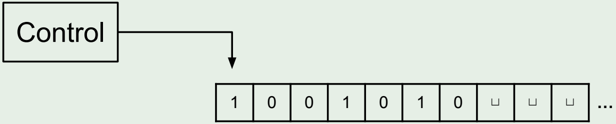

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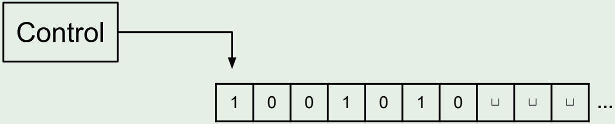

Figure: Example Turing Machine

1. Scan input left to right
2. Replace the first 1 with an "x" and the first 0 with an "x"
3. If the entire string is full of x's, accept
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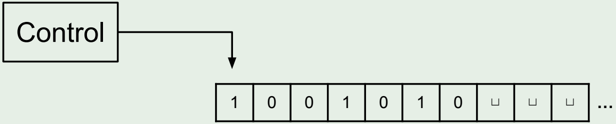

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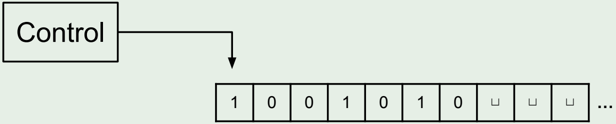

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6. Go to stage 1

Example

Let's see what would happen after the first pass of our algorithm

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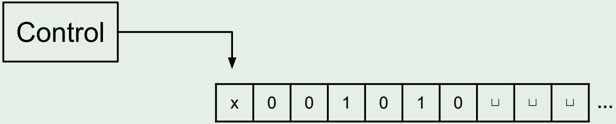

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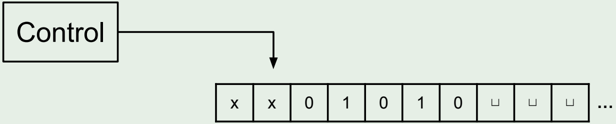

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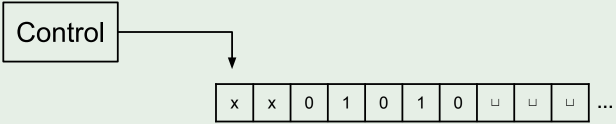

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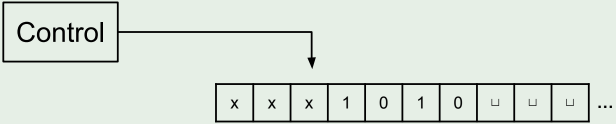

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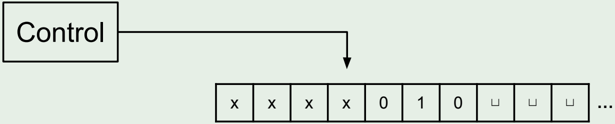

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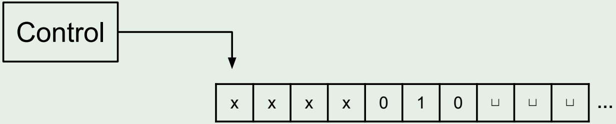

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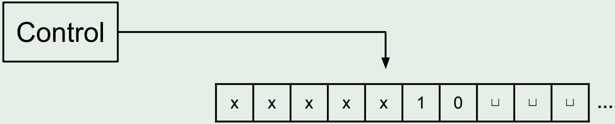

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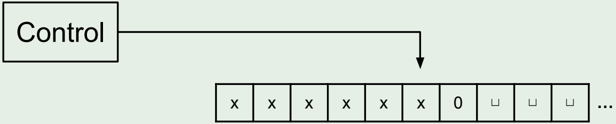

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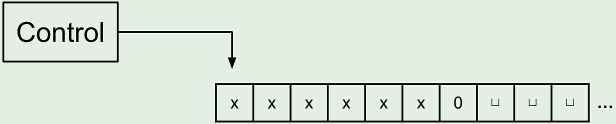

Figure: Example Turing Machine

Example

We see that at the end once we have filled out all the x's we possibly could we are left with only one non-x symbol, a 0, so we reject the string "1001010"

Example

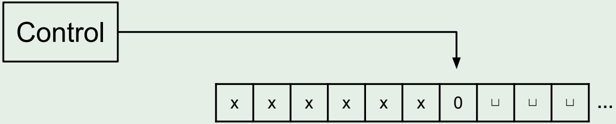

Figure: Example Turing Machine

Varients

What's next?

Varients

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There are many variants of Turing Machines, the ones which we will talk about today being the **Multitape Turing machine** and **Nondeterministic Turing Machine**. So far, we have only seen **Deterministic Turing Machines**.

Multitape Turing Machine

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A Multitape Turing Machine...

1. Exactly like Deterministic Turing Machine except...

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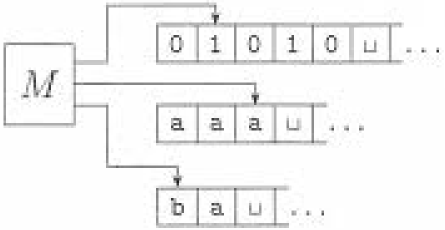

Figure: Multitape TM M

Theorem

Theorem

A Deterministic Turing Machine is equivalent to a Multitape Turing Machine

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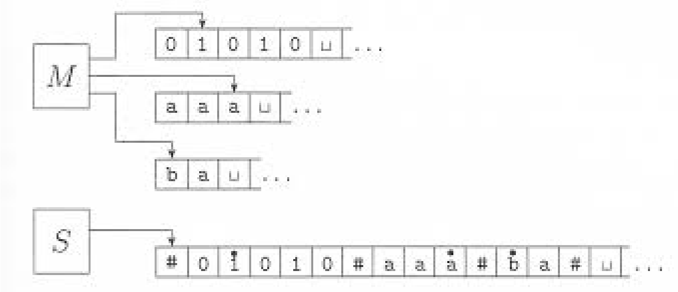

Figure: Multitape TM M and Deterministic TM S simulating it

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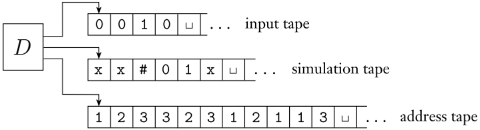

Figure: Nondeterministic TM being simulated on a Multitape TM

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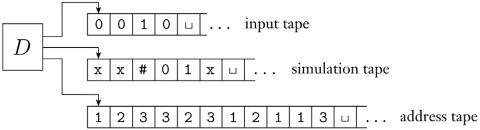

Figure: Nondeterministic TM being simulated on a Multitape TM

Deterministic TM = Multitape TM = Nondeterministic TM

Similarities

So what is the relationship between Turing Machines and Finite Automata?

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Ultimately for this reason Turing Machines are a lot more powerful than their finite counterparts.

Hierarchy

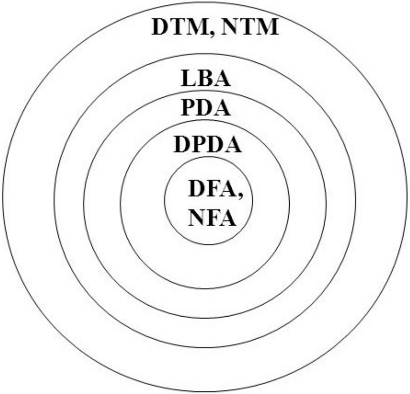

Figure: Hierarchy of Automata

Thank you!

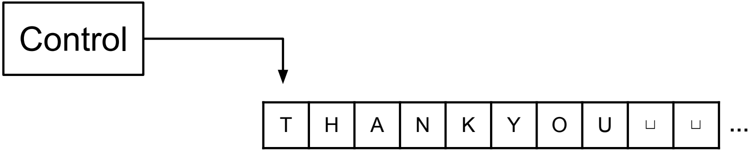

Figure: Thank you!